

# $\mathcal{N} = 2^*$ hydrodynamics

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## Abstract

Using gauge theory /string theory correspondence certain universal aspects of the strongly coupled four dimensional gauge theory hydrodynamics were established in hep-th/0311175. The analysis were performed in the framework of “membrane paradigm” approach to the fluctuations on the black brane stretched horizon. We confirm the universal result for the shear viscosity to the entropy density ratio for the strongly coupled  $\mathcal{N} = 2^*$  gauge theory from explicit computation of the finite temperature Minkowski-space correlation functions in the dual supergravity geometry.

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# 1 Introduction

Gauge theory/string theory correspondence [1] represents a valuable tool for analyzing dynamics of gauge theories. In particular, it provides an effective description of finite temperature strongly coupled gauge theories in terms of supergravity black hole backgrounds. Recently, with the formulation of the prescription for the computation of Minkowski-space correlation functions in gauge/gravity correspondence [2, 3], a study of *non-equilibrium* processes (*e.g.* diffusion and sound propagation) in gauge theory plasma became possible [4–8]. There is a hope that dual supergravity analysis of plasma transport coefficients will be of utility in hydrodynamic models used to describe elliptic flows in heavy ion collision experiments at RHIC [9–11].

From the fundamental perspective, an intriguing spin-off of the strongly coupled gauge theory hydrodynamics analysis was the formulation by Kovtun, Son and Starinets (KSS) [12] the gauge theory shear viscosity  $\eta$  bound in terms of its entropy density  $s$

$$\frac{\eta}{s} \geq \frac{\hbar}{4\pi k_B} \approx 6.08 \times 10^{-13} \text{ K} \cdot \text{s}. \quad (1.1)$$

Specifically, in the framework of “membrane paradigm” [13, 14] approach to the fluctuations on black brane stretched horizon, KSS established saturation of the viscosity bound (1.1) for all maximally supersymmetric gauge theories and for  $\mathcal{N} = 2^*$  gauge theory (to leading order in  $m/T$ ) [15, 16]. In the case of maximally supersymmetric gauge theories, “membrane paradigm” computations were shown to reproduce analysis of the transport coefficients extracted from Minkowski-space correlators. Moreover, since the near horizon black brane geometries dual to finite temperature maximally supersymmetric gauge theories allow for an extension to asymptotically flat space-times, the universality of  $\eta/s$  can be related [17, 18] to the universality of low energy absorption cross sections for black holes observed in [19]. We emphasize that existence of the asymptotically flat region is absolutely crucial to establish the latter connection, as only in that case one can set up a graviton scattering “experiment” of [19].

Membrane paradigm framework toward gauge theory hydrodynamics [12], though lacking a rigorous physical understanding enjoyed within correlation function approach [5], is ultimately the most flexible (and the easiest to implement). For one reason, it does not rely on the existence of the asymptotically flat region<sup>1</sup>. Second, such computations are sensitive only to the local geometry in the vicinity of the horizon, and thus

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<sup>1</sup>Unlike maximally supersymmetric examples of the gauge/gravity correspondence, asymptotically flat extensions of the supergravity backgrounds dual to non-conformal gauge theories are not known.

following the gauge theory/ string theory correspondence should describe the *infrared* properties of the dual gauge theory. This fits neatly with the viewpoint of hydrodynamics as a *low-energy* effective description of the system close to equilibrium. Also, this makes an observation that supergravity typically realize standard four-dimensional non-conformal gauge theory only in the IR (as in [20, 21]) irrelevant. In fact, correlation function approach toward computation of the transport coefficients for the gauge theories dual to geometries of [20, 21] is rather subtle, as it is linked to (yet unresolved) issues of local regularization of such models [22]. Within membrane paradigm approach, a theorem was proved [23] that all systems, admitting a supergravity holographic dual (with a low-energy gauge theory description) saturate (1.1) at infinite 't Hooft coupling<sup>2</sup>:

$$\begin{aligned}\frac{\eta}{s} &= f\left(g_{YN}^2 N, \frac{\Lambda_i}{T}\right) \frac{1}{4\pi}, \\ \lim_{g_{YN}^2 N \rightarrow \infty} f\left(g_{YN}^2 N, \frac{\Lambda_i}{T}\right) &= 1,\end{aligned}\tag{1.2}$$

independent of any microscopic scales  $\{\Lambda_i\}$  relative to the temperature  $T$ .

Given a somewhat surprising conclusion (1.2), and a conjectural status of the membrane paradigm approach, we feel independent verification of (1.2) for non-conformal gauge theories is highly desirable. Such a verification is provided here via explicit computation of the Minkowski-space correlation functions in the supergravity dual to finite temperature  $\mathcal{N} = 2^*$  gauge theory. Though the relevant black hole geometry [16] is not known analytically, we will be able to perform analytical analysis of the shear mode fluctuations using two complimentary approaches. First, we compute shear viscosity using Kubo formula from the correlation function of the stress-energy tensor at zero spatial momentum

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d\bar{x} e^{i\omega t} \langle [T_{xy}(x), T_{xy}(0)] \rangle.\tag{1.3}$$

We find that though

$$\eta = \eta\left(T, \frac{m_b}{T}, \frac{m_f}{T}\right),\tag{1.4}$$

where [16]  $m_b$  ( $m_f$ ) are (generically different) masses of the bosonic (fermionic) components of the  $\mathcal{N} = 2^*$  hypermultiplet,

$$\frac{\eta}{s} = \frac{1}{4\pi},\tag{1.5}$$

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<sup>2</sup>Finite 't Hooft coupling corrections (dual to  $\alpha'$ -corrections on the string theory side of the correspondence) to (1.2) are nonvanishing, and will be reported in [24].

independent of the conformal symmetry breaking scales  $m_b, m_f$ . Second, from the stress-tensor correlation functions which have a diffusion pole, we directly extract the shear diffusion constant  $\mathcal{D}$

$$\mathcal{D} = \frac{1}{4\pi T}, \quad (1.6)$$

which given identity

$$\frac{\eta}{s} = \mathcal{D}T, \quad (1.7)$$

reproduces Kubo formula result (1.5).

Of course, prior to computation of the correlation functions, the thermodynamics of the supergravity background must be understood. While the non-extremal deformation of the PW flow [15] was constructed in [16], the complete understanding of its thermodynamics was lacking. Using [22], we resolve the puzzle of the black hole thermodynamics of [16].

## 2 $\mathcal{N} = 2^*$ thermodynamics

### 2.1 The geometry

The supergravity background dual to finite temperature  $\mathcal{N} = 2^*$  gauge theory was studied in [16]. Here we collect the relevant facts about the geometry referring for the details to the original analysis.

The effective five-dimensional action is

$$\begin{aligned} S &= \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \mathcal{L}_5 \\ &= \frac{1}{4\pi G_5} \int_{\mathcal{M}_5} d\xi^5 \sqrt{-g} \left( \frac{1}{4}R - 3(\partial\alpha)^2 - (\partial\chi)^2 - \mathcal{P} \right), \end{aligned} \quad (2.1)$$

where the potential  $\mathcal{P}$  is<sup>3</sup>

$$\mathcal{P} = \frac{1}{16} \left[ \frac{1}{3} \left( \frac{\partial W}{\partial \alpha} \right)^2 + \left( \frac{\partial W}{\partial \chi} \right)^2 \right] - \frac{1}{3} W^2, \quad (2.2)$$

with the superpotential

$$W = -\frac{1}{\rho^2} - \frac{1}{2} \rho^4 \cosh(2\chi). \quad (2.3)$$

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<sup>3</sup>We set the 5d gauged supergravity coupling to one. This corresponds to setting the  $S^5$  radius  $L = 2$ .

The five dimensional Newton's constant is

$$G_5 \equiv \frac{G_{10}}{2^5 \text{vol}_{S^5}} = \frac{4\pi}{N^2}. \quad (2.4)$$

The action (2.1) yields the Einstein equation

$$\frac{1}{4}R_{\mu\nu} = 3\partial_\mu\alpha\partial_\nu\alpha + \partial_\mu\chi\partial_\nu\chi + \frac{1}{3}g_{\mu\nu}\mathcal{P}, \quad (2.5)$$

and the scalar equations

$$\square\alpha = \frac{1}{6}\frac{\partial\mathcal{P}}{\partial\alpha}, \quad \square\chi = \frac{1}{2}\frac{\partial\mathcal{P}}{\partial\chi}. \quad (2.6)$$

For a finite temperature deformation of the PW [15] flow metric we take

$$ds_5^2 = e^{2A} (-e^{2B} dt^2 + d\vec{x}^2) + dr^2, \quad (2.7)$$

where  $e^{2B}$  represents a blackening function. Note that we choose to retain  $g_{rr} = 1$  since any non-trivial factor can be absorbed into a redefinition of  $r$ .

Substituting this metric ansatz into the equations of motion, (2.5) and (2.6), we find

$$\begin{aligned} 0 &= \alpha'' + (4A' + B')\alpha' - \frac{1}{6}\frac{\partial\mathcal{P}}{\partial\alpha}, \\ 0 &= \chi'' + (4A' + B')\chi' - \frac{1}{2}\frac{\partial\mathcal{P}}{\partial\chi}, \\ 0 &= B'' + (4A' + B')B', \\ \frac{1}{4}A'' + \frac{1}{4}B'' + (A')^2 + \frac{1}{4}(B')^2 + \frac{5}{4}A'B' &= -\frac{1}{3}\mathcal{P}, \\ -A'' - \frac{1}{4}B'' - (A')^2 - \frac{1}{4}(B')^2 - \frac{1}{2}A'B' &= 3(\alpha')^2 + (\chi')^2 + \frac{1}{3}\mathcal{P}. \end{aligned} \quad (2.8)$$

Notice that the equation for  $B$  in (2.8) can be integrated once to obtain

$$\ln B' + 4A + B = \text{const}. \quad (2.9)$$

This relation will prove useful below.

Nonsingular in the IR flows of (2.8) are given by a three parameter family  $\{\alpha, \rho_0 >$

$0, \chi_0\}$ , specifying the near horizon ( $r \rightarrow 0$ ) Taylor series expansions

$$\begin{aligned} e^A &= e^\alpha \left[ 1 + \left( \sum_{i=1}^{\infty} a_i r^{2i} \right) \right], \\ e^B &= \delta r \left( 1 + \sum_{i=1}^{\infty} b_i r^{2i} \right), \\ \rho &= \rho_0 + \left( \sum_{i=1}^{\infty} \rho_i r^{2i} \right), \\ \chi &= \chi_0 + \left( \sum_{i=1}^{\infty} \chi_i r^{2i} \right). \end{aligned} \tag{2.10}$$

Here,  $\delta = \delta(\rho_0, \chi_0)$  should be adjusted so that  $e^B \rightarrow 1_-$  as  $r \rightarrow +\infty$ . The first non-trivial terms in the series expansions (2.10) are

$$\begin{aligned} \delta^{-2} a_1 &= \frac{1}{12} \rho_0^{-4} + \frac{1}{6} \rho_0^2 \cosh(2\chi_0) - \frac{1}{48} \rho_0^8 \sinh^2(2\chi_0), \\ \delta^{-2} b_1 &= -\frac{1}{9} \rho_0^{-4} - \frac{2}{9} \rho_0^2 \cosh(2\chi_0) + \frac{1}{36} \rho_0^8 \sinh^2(2\chi_0), \\ \delta^{-2} \rho_1 &= \frac{1}{24} \rho_0^{-3} - \frac{1}{24} \rho_0^3 \cosh(2\chi_0) + \frac{1}{48} \rho_0^9 \sinh^2(2\chi_0), \\ \delta^{-2} \chi_1 &= -\frac{1}{8} \rho_0^2 \sinh(2\chi_0) + \frac{1}{64} \rho_0^8 \sinh(4\chi_0). \end{aligned} \tag{2.11}$$

Three integration constants  $\{\alpha, \chi_0, \rho_0\}$  are related to temperature and masses of the  $\mathcal{N} = 2$  hypermultiplet components. The most general solution of (2.8) in the UV ( $\chi \rightarrow 0_+$ ) has altogether five parameters,  $\{\xi, \hat{\rho}_{10}, \hat{\rho}_{11}, \hat{\chi}_0, \hat{\chi}_{10}\}$ . Three of them are related to the temperature and the masses, while the other two are uniquely determined from the requirement of having a regular horizon, (2.11). In any case, we have a three parameter BH solution<sup>4</sup>

$$\begin{aligned} B \sim & -\beta x^4 \left[ 1 + \frac{8}{9} x^2 \hat{\chi}_0^2 + x^4 \left( \frac{5}{16} \hat{\rho}_{11}^2 - \frac{1}{2} \hat{\rho}_{11} \hat{\rho}_{10} + \frac{1}{18} \hat{\chi}_0^4 + 2 \hat{\rho}_{10}^2 + \hat{\chi}_0^2 \hat{\chi}_{10} \right. \right. \\ & \left. \left. + \ln x \left( -\frac{1}{2} \hat{\rho}_{11}^2 + \frac{4}{3} \hat{\chi}_0^4 + 4 \hat{\rho}_{11} \hat{\rho}_{10} \right) + 2 \hat{\rho}_{11}^2 \ln^2 x \right) \right], \end{aligned} \tag{2.12}$$

$$\begin{aligned} \chi \sim & \hat{\chi}_0 x \left[ 1 + x^2 \left( \hat{\chi}_{10} + \frac{4}{3} \hat{\chi}_0^2 \ln x \right) + x^4 \left( \frac{31}{8} \hat{\rho}_{11}^2 - \frac{13}{2} \hat{\rho}_{11} \hat{\rho}_{10} - \frac{56}{45} \hat{\chi}_0^4 - \frac{3}{2} \hat{\chi}_0^2 \hat{\rho}_{11} + 2 \hat{\chi}_0^2 \hat{\rho}_{10} \right. \right. \\ & \left. \left. + 5 \hat{\rho}_{10}^2 + 2 \hat{\chi}_0^2 \hat{\chi}_{10} + \ln x \left( -\frac{13}{2} \hat{\rho}_{11}^2 + 10 \hat{\rho}_{11} \hat{\rho}_{10} + \frac{8}{3} \hat{\chi}_0^4 + 2 \hat{\chi}_0^2 \hat{\rho}_{11} \right) + 5 \hat{\rho}_{11}^2 \ln^2 x \right) \right], \end{aligned} \tag{2.13}$$

$$\begin{aligned} \rho \sim & 1 + x^2 \left( \hat{\rho}_{10} + \hat{\rho}_{11} \ln x \right) + x^4 \left( -2 \hat{\rho}_{11} \hat{\rho}_{10} + \frac{3}{2} \hat{\rho}_{11}^2 + \frac{3}{2} \hat{\rho}_{10}^2 + \frac{10}{3} \hat{\chi}_0^2 \hat{\rho}_{10} - \frac{8}{3} \hat{\chi}_0^2 \hat{\rho}_{11} + \frac{1}{3} \hat{\chi}_0^4 \right. \\ & \left. + \ln x \left( 3 \hat{\rho}_{11} \hat{\rho}_{10} + \frac{10}{3} \hat{\chi}_0^2 \hat{\rho}_{11} - 2 \hat{\rho}_{11}^2 \right) + \frac{3}{2} \hat{\rho}_{11}^2 \ln^2 x \right), \end{aligned} \tag{2.14}$$

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<sup>4</sup>The integration constant  $\beta$  can be absorbed at the expense of shifting the position of the horizon in the radial coordinate  $r$ , or alternatively by rescaling  $x$ .

$$A \sim \xi - \ln x - \frac{1}{3}\hat{\chi}_0^2 x^2 + x^4 \left( \frac{1}{4}\beta + \frac{1}{9}\hat{\chi}_0^4 - \frac{1}{2}\hat{\chi}_0^2 \hat{\chi}_{10} - \frac{1}{8}\hat{\rho}_{11}^2 - \hat{\rho}_{10}^2 \right. \\ \left. - \ln x \left( \frac{2}{3}\hat{\chi}_0^4 + 2\hat{\rho}_{11}\hat{\rho}_{10} \right) - \hat{\rho}_{11}^2 \ln^2 x \right). \quad (2.15)$$

Also, we find

$$\frac{dA}{dr} \sim \frac{1}{2} + \frac{1}{3}\hat{\chi}_0^2 x^2 + x^4 \left( -\frac{1}{2}\beta + 2\hat{\rho}_{10}^2 + \hat{\rho}_{11}\hat{\rho}_{10} + \frac{1}{4}\hat{\rho}_{11}^2 + \hat{\chi}_0^2 \hat{\chi}_{10} + \frac{1}{9}\hat{\chi}_0^4 \right. \\ \left. + \ln x \left( \hat{\rho}_{11}^2 + \frac{4}{3}\hat{\chi}_0^4 + 4\hat{\rho}_{11}\hat{\rho}_{10} \right) + 2\hat{\rho}_{11}^2 \ln^2 x \right). \quad (2.16)$$

In (2.12)-(2.16),  $x = x_0 e^{-r/2}$  with  $x_0$  an arbitrary constant.

It is straightforward to compute the Bekenstein-Hawking entropy density

$$s = \frac{\mathcal{A}_{horizon}}{4G_N} = \frac{1}{2} \pi^2 N^2 \left( \frac{1}{2\pi} e^\alpha \right)^3, \quad (2.17)$$

and the black hole temperature

$$T = \frac{1}{2\pi} e^A \left( \frac{\partial e^B}{\partial r} \right) \Big|_{r \rightarrow 0_+} = \frac{1}{2\pi} e^\alpha \delta. \quad (2.18)$$

## 2.2 Boundary renormalization

Understanding the black hole thermodynamics mandates understanding the corresponding geometry energy (ADM mass) and the free energy. Following ideas of [27–29], the ADM mass can be computed as a one-point correlation function of the boundary stress tensor, while the free energy can be extracted from the expectation value of the Euclidean gravitation action. Both quantities are infinite and must be properly regularized and renormalized<sup>5</sup>. Below, we present the necessary results while referring for details to [22].

Let  $r$  be the position of the boundary, and  $S_E^r$  be the Euclidean gravitational action on the cut-off space

$$\lim_{r \rightarrow \infty} S_E^r = S_E, \quad (2.19)$$

where  $S_E$  is the Euclidean version of (2.1). Besides the standard Gibbons-Hawking term

$$S_{GH} = -\frac{1}{8\pi G_5} \int_{\partial \mathcal{M}_5} d^4x \sqrt{h_E} \nabla_\mu n^\mu = -\frac{1}{8\pi G_5} \left[ (e^{4A+3B})' \right] \Big|_{\partial \mathcal{M}_5}^r \int_{\partial \mathcal{M}_5} d^4x, \quad (2.20)$$

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<sup>5</sup>Understanding the renormalization of the gravitation action is also vital for the computation of the correlation functions in the framework of gauge theory / string theory correspondence.

we supplement the combined regularized action ( $S_E^r + S_{GH}$ ) by the appropriate boundary counterterms which are needed to get a finite action. These boundary counterterms must be constructed from the local metric and  $\{\alpha = \ln \rho, \chi\}$  scalar invariants on the boundary  $\partial\mathcal{M}_5$ , except for the terms associated with the conformal anomaly which include an explicit dependence on the position of the boundary,

$$S^{counter} = \frac{1}{4\pi G_5} \int_{\partial\mathcal{M}_5} d^4x \sqrt{h_E} \left( \alpha_1 + \alpha_2 R_4 + \alpha_3 \alpha + \alpha_4 \chi^2 + \alpha_5 \alpha^2 + \alpha_6 \alpha \chi^2 + \alpha_7 R_4 \alpha + \alpha_8 \frac{\alpha^2}{\ln \epsilon} + \ln \epsilon (\alpha_9 R_4 \chi^2 + \alpha_{10} \chi^4) + \alpha_{11} \chi^4 \right). \quad (2.21)$$

Here  $R_4 \equiv R_4(h_E)$  is the Ricci scalar constructed from  $h_{\mu\nu}$  and the  $\alpha_i$  are constant coefficients of the counterterms which are determined by the requirement of having a finite action (the coefficients  $\alpha_8, \alpha_9, \alpha_{10}$  which are associated with the conformal anomaly have been computed before but we leave them arbitrary for now). The quartic term in  $\chi$  is actually finite and is introduced to preserve supersymmetry. The conformal anomaly terms depend on the position of the boundary; we choose to parameterize this position by the physical quantity [22]

$$\epsilon \equiv \sqrt{-g_{tt}}. \quad (2.22)$$

The counterterm parameters  $\alpha_i$  are fixed in such a way that the *renormalized* Euclidean action  $I_E$  is finite

$$I_E \equiv \lim_{r \rightarrow \infty} \left( S_E^r + S_{GH} + S^{counter} \right), \quad |I_E| < \infty. \quad (2.23)$$

We find [22]

$$\begin{aligned} \alpha_1 &= \frac{3}{4}, & \alpha_2 &= \frac{1}{4}, & \alpha_3 &= 0, & \alpha_4 &= \frac{1}{2}, & \alpha_5 &= 3, & \alpha_6 &= 0, \\ \alpha_7 &= 0, & \alpha_8 &= -\frac{3}{2}, & \alpha_9 &= -\frac{1}{3}, & \alpha_{10} &= -\frac{2}{3}, & \alpha_{11} &= \frac{1}{6}. \end{aligned} \quad (2.24)$$

The quasilocal stress tensor  $T_{\mu\nu}$  for our background is obtained from the variation of the full action

$$S_{tot} = S_E^r + S_{GH} + S^{counter}, \quad (2.25)$$

with respect to the boundary metric  $\delta h_{\mu\nu}$

$$T^{\mu\nu} = \frac{2}{\sqrt{-h}} \frac{\delta S_{tot}}{\delta h_{\mu\nu}}. \quad (2.26)$$



Explicit computation yields

$$\begin{aligned}
T^{\mu\nu} = & \frac{1}{8\pi G_5} \left[ -\Theta^{\mu\nu} + \Theta h^{\mu\nu} \right. \\
& - 2 \left\{ \alpha_1 + \alpha_3 \alpha + \alpha_4 \chi^2 + \alpha_5 \alpha^2 + \alpha_6 \alpha \chi^2 + \alpha_8 \frac{\alpha^2}{\ln \epsilon} + \alpha_{10} \ln \epsilon \chi^4 + \alpha_{11} \chi^4 \right\} h^{\mu\nu} \\
& \left. + 4 \left\{ \alpha_2 + \alpha_7 \alpha + \alpha_9 \ln \epsilon \chi^2 \right\} \left( R_4^{\mu\nu} - \frac{1}{2} R_4 h^{\mu\nu} \right) \right],
\end{aligned}
\tag{2.27}$$

where

$$\Theta^{\mu\nu} = \frac{1}{2} (\nabla^\mu n^\nu + \nabla^\nu n^\mu), \quad \Theta = \text{Tr } \Theta^{\mu\nu}.
\tag{2.28}$$

### 2.3 The thermodynamics

The thermodynamics of the PW flow [15] has been studied before in [16]. In [16] computation of the free energy and the energy (ADM mass) has been done using the background subtraction prescription of [25], with a reference background being the supersymmetric PW flow. Though one can regularize in this way the free energy  $F$ , the energy  $E$ , prove the relation

$$F = E - TS,
\tag{2.29}$$

where  $S$ ,  $T$  are the entropy and the Hawking temperature of the nonextremal PW deformation, one finds

$$T dS \neq dE.
\tag{2.30}$$

The disagreement with the first law of thermodynamics either points to incorrect subtraction prescription, or to a more exotic explanation, like a chemical potential [16]. That subtraction prescription [25] does not always work is well known. In fact, in some cases one simply can not find an appropriate reference geometry [26]. What is surprising with non-extremal deformation of the PW flow, is the fact that a well-motivated reference background leads to a problematic thermodynamics. An alternative prescription of regularizing the free energy and the ADM mass of a gravity background dual to a gauge theory (in a sense of [1]) has been formulated in [27], and applied to an example where approach of [25] can not work, in [26]. With understanding of the renormalization of the gravity dual to  $\mathcal{N} = 2^*$  gauge theory in previous section, we can study its thermodynamics following [27, 30, 26]. In this section we show that

with the counterterm structure (2.21), (2.24) the first law of thermodynamics for the nonextremal deformation of the PW solution [16] holds. We follow mostly [16]. Note the difference in normalization of a 5d scalar  $\alpha$  here and in [16],  $\alpha_{BL}$

$$\alpha_{BL} \equiv \sqrt{3}\alpha, \quad \rho_{BL} \equiv \rho. \quad (2.31)$$

As in [16] we identify the regularized Euclidean action with  $F/T$

$$F \equiv T I_E, \quad (2.32)$$

and the energy (ADM mass)  $E$  with

$$E = \int_{\Sigma} d^3\xi \sqrt{\sigma} N_{\Sigma} \mathcal{E}, \quad (2.33)$$

where<sup>6</sup>  $\Sigma \equiv R^3$  is a spacelike hypersurface in  $\partial\mathcal{M}_5$  with a timelike unit normal  $u^\mu$ ,  $N_{\Sigma}$  is the norm of the timelike Killing vector in (2.7),  $\sigma$  is the determinant of the induced metric on  $\Sigma$ , and  $\mathcal{E}$  is the proper energy density

$$\mathcal{E} = u^\mu u^\nu T_{\mu\nu}. \quad (2.34)$$

The quasilocal stress tensor  $T_{\mu\nu}$  is computed following (2.26). Combining eqs. (6.18), (6.22) of [16] with the relevant terms of (2.21) we find the equivalent of (2.23)

$$\begin{aligned} I_E &= \frac{1}{8\pi G_5} \frac{V_3}{T} \left( -e^{3A} (e^{A+B})' \Big|_{horizon} + \lim_{r \rightarrow \infty} \left[ -3e^{4A+B} A' + 2e^{4A+B} \left\{ \alpha_1 + \alpha_3 \alpha \right. \right. \right. \\ &\quad \left. \left. \left. + \alpha_4 \chi^2 + \alpha_5 \alpha^2 + \alpha_6 \alpha \chi^2 + \alpha_8 \frac{\alpha^2}{\ln \epsilon} + \ln \epsilon \alpha_{10} \chi^4 + \alpha_{11} \chi^4 \right\} \right] \right) \\ &\equiv \frac{1}{8\pi G_5} \frac{V_3}{T} \left( -e^{3A} (e^{A+B})' \Big|_{horizon} + \Delta_{BH} \right) \\ &= -S + \frac{V_3 \Delta_{BH}}{8\pi G_5 T}, \end{aligned} \quad (2.35)$$

where  $S$  is the entropy of the black hole horizon, and we defined  $\Delta_{BH}$  as above. Additionally, as in (2.22) we identify

$$\epsilon \equiv \sqrt{-g_{tt}} = e^{A+B}. \quad (2.36)$$

Also

$$T_{tt} = \frac{h_{tt}}{8\pi G_5} \left[ -3A' + 2 \left\{ \alpha_1 + \alpha_3 \alpha + \alpha_4 \chi^2 + \alpha_5 \alpha^2 + \alpha_6 \alpha \chi^2 + \alpha_8 \frac{\alpha^2}{\ln \epsilon} \right. \right. \\ \left. \left. + \ln \epsilon \alpha_{10} \chi^4 + \alpha_{11} \chi^4 \right\} \right]. \quad (2.37)$$

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<sup>6</sup>We take the volume of  $R^3$  to be  $V_3$ .

From (2.33), (2.37) we have

$$\begin{aligned}
E &= \frac{1}{8\pi G_5} V_3 \lim_{r \rightarrow \infty} \left[ -3e^{4A+B} A' + 2e^{4A+B} \left\{ \alpha_1 + \alpha_3 \alpha \right. \right. \\
&\quad \left. \left. + \alpha_4 \chi^2 + \alpha_5 \alpha^2 + \alpha_6 \alpha \chi^2 + \alpha_8 \frac{\alpha^2}{\ln \epsilon} + \ln \epsilon \alpha_{10} \chi^4 + \alpha_{11} \chi^4 \right\} \right] \\
&= \frac{V_3 \Delta_{BH}}{8\pi G_5} .
\end{aligned} \tag{2.38}$$

Notice that from (2.32), (2.35), (2.38) we trivially recover (2.29). The asymptotic solution for  $\{A, B, \rho, \chi\}$  is given in (2.12)-(2.16), using which we find that  $\Delta_{BH}$  is indeed finite with

$$\Delta_{BH} = -\frac{1}{12} e^{4\xi} \left( -18\beta + 9\hat{\rho}_{11}^2 + 12\hat{\chi}_0^2 \hat{\chi}_{10} - 36\hat{\rho}_{11} \hat{\rho}_{10} + 16\hat{\chi}_0^4 \xi - 36\hat{\rho}_{11}^2 \xi \right) , \tag{2.39}$$

where we used (2.24). As in [16], we can not study analytically the thermodynamics apart from the high temperature regime,  $\frac{m}{T} \ll 1$ , where  $m$  is the mass of the  $\mathcal{N} = 2$  hypermultiplet of the dual gauge theory. Using the leading order analytical solution of Sec. 5 of [16], specifically eqs. (6.40)-(6.45), we find

$$\begin{aligned}
S &= \frac{1}{2} \pi^2 N^2 T^3 \left( 1 - \frac{\Gamma(3/4)^4}{\pi^4} \frac{m^2}{T^2} \right) , \\
E &= \frac{3}{8} \pi^2 N^2 T^4 \left( 1 - \frac{2}{3} \frac{\Gamma(3/4)^4}{\pi^4} \frac{m^2}{T^2} \right) , \\
F &= -\frac{1}{8} \pi^2 N^2 T^4 \left( 1 - 2 \frac{\Gamma(3/4)^4}{\pi^4} \frac{m^2}{T^2} \right) .
\end{aligned} \tag{2.40}$$

with

$$T dS = dE . \tag{2.41}$$

### 3 The viscosity bound

In this section we closely follow [5]. To compute two point correlation functions of the boundary stress-energy tensor we consider small fluctuations of the near-extremal PW flow metric

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + h_{\mu\nu} , \tag{3.1}$$

where  $g_{\mu\nu}^{(0)}$  is the background metric (2.7). As in [5], we assume that metric perturbations take the form

$$h_{\mu\nu} = e^{-i\omega t + i q z} H_{\mu\nu}(r) . \tag{3.2}$$

Residual  $O(2)$  symmetry of rotations in  $xy$  plane guarantees that (at a linearized level) field equations for  $\{h_{xy}\}$  and  $\{h_{tx}, h_{xz}\}$  (as well as the other metric components) decouple. We use  $h_{xy}$  perturbations for the computation of the  $T_{xy}T_{xy}$  correlation functions to be used in Kubo formula (1.3); the (coupled) fluctuations  $\{h_{tx}, h_{xz}\}$  will produce correlation functions with a diffusion pole.

### 3.1 Shear viscosity via Kubo relation

Eq. (1.3) can be written in the form

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega i} \left[ G_{xy,xy}^A(\omega, 0) - G_{xy,xy}^R(\omega, 0) \right], \quad (3.3)$$

where the retarded Green's function is defined as

$$G_{\mu\nu,\lambda\rho}^R(\omega, \bar{q}) = -i \int d^4x e^{-iq \cdot x} \theta(t) \langle [T_{\mu\nu}(x), T_{\lambda\rho}(0)] \rangle, \quad (3.4)$$

and  $G^A(\omega, \bar{q}) = (G^R(\omega, \bar{q}))^*$ .

Following [2,3], retarded correlation function  $G_{xy,xy}^R(\omega, \bar{q})$  can be extracted from the (quadratic) boundary effective action  $S_{boundary}$  for the metric fluctuations  $h_{xy}^b$

$$h_{xy}^b(k) \equiv h_{xy}^b(\omega, \bar{q}) = \int \frac{d^4k}{(2\pi)^4} e^{-i\omega t + iq \cdot x} h_{xy}(t, \bar{x}, r), \quad \text{as } r \rightarrow \infty, \quad (3.5)$$

given by

$$S_{boundary}[h_{xy}^b] = \int \frac{d^4k}{(2\pi)^4} h_{xy}^b(-k) \mathcal{F}(k, r) h_{xy}^b(k) \Big|_{r \rightarrow 0}^{r \rightarrow \infty}, \quad (3.6)$$

as

$$G_{xy,xy}^R(\omega, \bar{q}) = \lim_{r \rightarrow \infty} 8 \mathcal{F}(k, r). \quad (3.7)$$

The boundary metric functional is defined as

$$S_{boundary}[h_{xy}^b] = \lim_{r \rightarrow \infty} \left( S^r[h_{xy}] + S_{GH}[h_{xy}] + S^{counter}[h_{xy}] \right), \quad (3.8)$$

where  $S^r$  is the bulk Minkowski-space cut-off action (2.7), evaluated on-shell for the bulk metric fluctuations  $h_{xy}(t, \bar{x}, r)$  subject to the following boundary conditions:

$$\begin{aligned} (a) : \quad & \lim_{r \rightarrow \infty} h_{xy}(t, \bar{x}, r) = h_{xy}^b(t, \bar{x}), \\ (b) : \quad & h_{xy}(t, \bar{x}, r) \text{ is an incoming wave at the horizon } (r \rightarrow \infty). \end{aligned} \quad (3.9)$$

The purpose of the boundary counterterm  $S^{counter}$  (2.21) is to remove divergent (as  $r \rightarrow \infty$ ) and  $\{\omega, \bar{q}\}$ -independent contributions from the kernel  $\mathcal{F}$  of (3.6). Despite the absence of the analytical solution to (2.8) for generic bosonic and fermionic masses<sup>7</sup>

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<sup>7</sup>Leading in  $\{m_b/T, m_f/T\}$  solution was given in [16].

( $m_b$  and  $m_f$ ), we will be able to implement all steps outlined, necessary to extract  $\eta$  analytically.

Introducing<sup>8</sup>

$$h_{xy}(t, r) \equiv 2 e^{2A} \phi(t, r), \quad (3.10)$$

the effective bulk action for  $\phi(t, r)$  becomes that of a minimally coupled scalar in the background (2.7). Further introducing a new radial coordinate  $y$ :

$$y \equiv e^{B(r)}, \quad (3.11)$$

so that  $y \rightarrow 0_+$  corresponds to the horizon, and  $y \rightarrow 1_-$  corresponds to the boundary, and

$$\phi(t, r) = e^{-i\omega t} \phi_k(y), \quad (3.12)$$

we find

$$\begin{aligned} 0 = & y^2 \left( 1 + 4\rho^6 \cosh^2 \chi - 2\rho^6 - \rho^{12} \cosh^4 \chi + \rho^{12} \cosh^2 \chi \right) (y \phi_k'' + \phi_k') \\ & + 2e^{-2A} \omega^2 \rho^2 \left( 3 \rho^2 A' + 6 y \rho^2 (A')^2 - 6 y (\rho')^2 - 2 y \rho^2 (\chi')^2 \right) \phi_k, \end{aligned} \quad (3.13)$$

where  $\{A, \rho, \chi\}$  are functions of  $y$  satisfying equations equivalent to (2.8), also all derivatives are with respect to  $y$ . A low-frequency solution of (3.13) which is an incoming wave at the horizon, and which near the boundary satisfies

$$\lim_{y \rightarrow 1_-} \phi_k(y) = 1, \quad (3.14)$$

can be written as

$$\phi_k(y) = y^{-i\omega Q} \left( F_0(y) + i\omega Q F_\omega(y) + \mathcal{O}(\omega^2) \right), \quad (3.15)$$

where, given the Taylor series expansion near the horizon (2.10),

$$Q = \frac{1}{\delta} e^{-\alpha}, \quad (3.16)$$

and  $\{F_0, F_\omega\}$  satisfy very simple ODE's:

$$\begin{aligned} 0 = & F_0' + y F_0'', \\ 0 = & y F_\omega'' + F_\omega' - 2 F_0'. \end{aligned} \quad (3.17)$$

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<sup>8</sup>We dropped the spacial dependence in  $h_{xy}(t, \bar{x}, r) \equiv h_{xy}(t, r)$  as we will need  $G^R$  ( $G^A$ ) at zero spacial momentum (3.3).

The only nonsingular solution to (3.17) which also satisfies (3.14) is

$$F_0(y) = 1, \quad F_\omega(y) = 0. \quad (3.18)$$

Thus,

$$\phi(t, r) = e^{-i\omega t} e^{-i\omega Q B(r)} (1 + \mathcal{O}(\omega^2)). \quad (3.19)$$

Once the bulk fluctuations are on-shell (*i.e.*, satisfy equations of motion) the bulk gravitational Lagrangian becomes a total derivative. Indeed, we find (without dropping any terms)

$$16\pi G_5 \mathcal{L}_5 = \partial_t J^t + \partial_r J^r, \quad (3.20)$$

where

$$\begin{aligned} J^r &= \frac{3}{2} e^{4A(r)+B(r)} \phi(t, r) \frac{\partial \phi(t, r)}{\partial r} + e^{4A(r)+B(r)} \frac{\partial A(r)}{\partial r} \phi(t, r)^2, \\ J^t &= -\frac{3}{2} e^{2A(r)-B(r)} \phi(t, r) \frac{\partial \phi(t, r)}{\partial t}. \end{aligned} \quad (3.21)$$

Additionally, the Gibbons-Hawking term provides an extra contribution so that

$$J^r \rightarrow J^r - 2e^{4A(r)+B(r)} \phi(t, r) \frac{\partial \phi(t, r)}{\partial r}. \quad (3.22)$$

Given the general asymptotic solution (2.12)-(2.16), we find

$$G_{xy,xy}^R(\omega, 0) = -\frac{i\omega Q e^{4\xi} \beta}{8\pi G_5} + \mathcal{O}(\omega^2). \quad (3.23)$$

Eq. (2.9) implies (see also eq.(6.33) of [16])

$$\ln \beta + \ln 2 + 4\xi = 4\alpha + \ln \delta \equiv \kappa. \quad (3.24)$$

Combining (2.4), (3.16) and (3.24) we rewrite (3.23) as

$$G_{xy,xy}^R(\omega, 0) = -\frac{i\omega N^2 e^{3\alpha}}{64\pi^2} + \mathcal{O}(\omega^2), \quad (3.25)$$

which leads to (3.3)

$$\eta = \frac{N^2 e^{3\alpha}}{64\pi^2} = \frac{1}{4\pi} s, \quad (3.26)$$

where we recalled the expression for the entropy density (2.17). As claimed, (3.26) reproduces the saturation (1.2) for arbitrary  $\{m_b/T, m_f/T\}$ .

### 3.2 The shear diffusion pole

Introducing

$$\begin{aligned} h_{tx}(t, z, y) &= e^{-i\omega t + iqz} e^{2A(y)} H_t(y), \\ h_{xz}(t, z, y) &= e^{-i\omega t + iqz} e^{2A(y)} H_z(y), \end{aligned} \quad (3.27)$$

the coupled system of equations of motion for the diffusive pole channel metric fluctuations becomes

$$\begin{aligned} 0 &= H'_t + \frac{q}{\omega} y^2 H'_z, \\ 0 &= e^{-6A+2\kappa} (y H''_t - H'_t) - y (\omega q H_z + q^2 H_t), \\ 0 &= e^{-6A+2\kappa} y (y H''_z + H'_z) + \omega q H_t + \omega^2 H_z, \end{aligned} \quad (3.28)$$

where all derivatives are with respect to  $y$  defined in (3.11), and  $\kappa$  is an integration constant (3.24). Solving for  $H_z$  from the second equation in (3.28) and substituting the result into the first equation of (3.28) we find

$$\begin{aligned} 0 &= e^{-6A+2\kappa} y^2 G'' - e^{-6A+2\kappa} y (6y A' + 2i\omega Q - 1) G' \\ &\quad + (\omega^2 + 6i\omega Q y e^{-6A+2\kappa} A' - \omega^2 Q^2 e^{-6A+2\kappa} - y^2 q^2) G, \end{aligned} \quad (3.29)$$

where we further extracted the singular part of  $H'_t$  which corresponds to the incoming wave boundary condition at the horizon

$$H'_t = y^{-i\omega Q + 1} G(y). \quad (3.30)$$

Quite amazingly, analytical (regular as  $y \rightarrow 0_+$ ) solution to (3.29) to leading order in  $\{\omega, q^2\}$ , can be found. Specifically,

$$G(y) \equiv G_0(y) + i\omega Q G_\omega(y) + q^2 G_q(y) + \mathcal{O}(\omega^2, \omega q^2, q^4), \quad (3.31)$$

with

$$\begin{aligned} G_0(y) &= C, \\ G_\omega(y) &= C \int_0^y dx \frac{1 - e^{6A(x) - 6\alpha}}{x}, \\ G_q(y) &= \frac{C}{2} \int_0^y dx x e^{6A(x) - 2\kappa}, \end{aligned} \quad (3.32)$$

where we used (3.16) and an explicit expression for  $\kappa$  (3.24). Up to an integration constant  $C$ , solution (3.32) is unique. Taking the limit  $y \rightarrow 1_-$  of the second equation

in (3.28), and using the general asymptotic (2.15) we find  $C$  in terms of the boundary values  $H_t^0$  and  $H_z^0$ :

$$C = -\frac{q^2 H_t^0 + \omega q H_z^0}{i\omega Q e^{2\kappa-6\alpha} - \frac{1}{2}q^2}. \quad (3.33)$$

Notice that there is a pole in (3.33) at

$$i\omega = \mathcal{D}q^2, \quad (3.34)$$

with

$$\mathcal{D} = \frac{1}{2\delta} e^{-\alpha} = \frac{1}{4\pi T}, \quad (3.35)$$

where we used (2.18). As in [5], it is easy to show that the pole in  $C$  would result in the identical pole in the retarded correlation functions  $G_{tx,tx}^R$ ,  $G_{tx,xz}^R$ ,  $G_{xz,xz}^R$ .

## 4 Conclusion

We performed explicit computations of the Minkowski-space stress-tensor correlation functions for  $\mathcal{N} = 2^*$  gauge theories at infinite 't Hooft coupling and confirmed the universality result [16], proved withing membrane paradigm approach. These computations provide additional support that the KSS bound is saturated at infinite 't Hooft coupling irrespectively of the microscopic scales of the system. It is an interesting question as to the modification (if any) of (1.2) once nonvanishing thermodynamic parameters (*e.g.*, chemical potentials) are introduced. A large class of supergravity backgrounds relevant for such a study is known.

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